Solution for USAMTS Problem 4/4/15

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Problem: For each non-negative integer n define the function $f_n(x)$ by

$$f_n(x) = \sin^n(x) + \sin^n\left(x + \frac{2\pi}{3}\right) + \sin^n\left(x + \frac{4\pi}{3}\right)$$

for all real numbers x, where the sine fuctions use radians. The functions $f_n(x)$ can also be expressed as polynomials in $\sin(3x)$ with rational coefficients. For example,

$$f_0(x) = 3, \qquad f_1(x) = 0, \qquad f_2(x) = \frac{3}{2}, \qquad f_3(x) = -\frac{3}{4}\sin(3x),$$
$$f_4(x) = \frac{9}{8}, \qquad f_5(x) = -\frac{15}{16}\sin(3x), \qquad f_6(x) = \frac{27}{32} + \frac{3}{16}\sin^2(3x),$$

for all real numbers x. Find an expression for $f_7(x)$ as a polynomial in $\sin(3x)$ with rational coefficients, and prove that it holds for all real numbers x.

Solution: Define

$$a = \sin(x), \qquad b = \sin\left(x + \frac{2\pi}{3}\right), \qquad c = \sin\left(x + \frac{4\pi}{3}\right),$$

so that $f_n(x) = a^n + b^n + c^n$. First, we have

$$a + b + c = f_1(x) = 0.$$

From the algebraic identity $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$, we find that

$$f_1(x)^2 = f_2(x) + 2(ab + bc + ca)$$
$$ab + bc + ca = \frac{1}{2}\left(f_1(x)^2 - f_2(x)\right)$$
$$ab + bc + ca = -\frac{3}{4}.$$

Finally, from the identity $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$,

$$f_3(x) - 3abc = f_1(x)\left(f_2(x) + \frac{3}{4}\right)$$
$$abc = \frac{1}{3}\left(f_3(x) - f_1(x)\left(f_2(x) + \frac{3}{4}\right)\right)$$
$$abc = -\frac{1}{4}\sin(3x).$$

When t equals a, b, or c, the quantity

$$\begin{aligned} F(t) &= (t-a)(t-b)(t-c) \\ &= t^3 - (a+b+c)t^2 + (ab+bc+ca)t - abc \\ &= t^3 - \frac{3}{4}t + \frac{1}{4}\sin(3x) \end{aligned}$$

is identically 0. Specifically, we get the three following equations:

$$\begin{aligned} a^n F(a) &= a^{n+3} - \frac{3}{4}a^{n+1} + \frac{1}{4}a^n \sin(3x) = 0 \\ b^n\, F(b) &= \ b^{n+3} - \frac{3}{4}b^{n+1} + \frac{1}{4}b^n \sin(3x) = 0 \\ c^n\, F(c) &= \ c^{n+3} - \frac{3}{4}c^{n+1} + \frac{1}{4}c^n \sin(3x) = 0 \end{aligned}$$

Adding the three previous equations gives a marvelous property of our function:

$$f_{n+3}(x) = \frac{3}{4}f_{n+1}(x) - \frac{1}{4}f_n(x)\sin(3x) \tag{1}$$

Using this recursion, the answer to the problem is immediate:

$$\begin{aligned} f_7(x) &= \frac{3}{4}f_5(x) - \frac{1}{4}f_4(x)\sin(3x) \\ &= -\frac{45}{64}\sin(3x) - \frac{9}{32}\sin(3x) \\ &= -\frac{63}{64}\sin(3x). \end{aligned}$$

Notice that the recursion given in equation (1) proves that $f_n(x)$ is a polynomial in $\sin(3x)$ with rational coefficients for all natural n.