On the Existence and Uniqueness of Invariant Measures on Locally Compact Groups

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1 Motivation and History

One of the most useful properties of $\mathbb{R}^n$ is invariance under a linear transformation. That is, if $a \in \mathbb{R}^n$ and f is any Lebesgue–integrable function, then

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{R}^n} f(y+a)\,dy.$$

Similarly, if we consider the multiplicative group of positive real numbers, $\mathbb{R}_+^\times$, and let k be a positive real number and f a Lebesgue–integrable function, then

$$\int_{\mathbb{R}_+^\times} f(x)\,\frac{dx}{x} = \int_{\mathbb{R}_+^\times} f(ky)\,\frac{dy}{y}.$$

The notion of Haar measure is a generalization of the above two examples. It turns out that in any locally compact group G, there exists a measure μ such that

$$\int_G f(x)\,d\mu(x) = \int_G f(gx)\,d\mu(x)$$

for any integrable function f and any $g \in G$.

At some time in the early twentieth century, people started to wonder if there was an invariant measure on all topological groups. The first two people to make significant progress on this problem were Alfréd Haar and John von Neumann in 1933. Haar in 1933 proved that there exists an invariant measure on any separable compact group.

Using Haar's result, von Neumann proved the special case of David Hilbert's Fifth Problem for compact locally Euclidean groups. The following year, von Neumann proved the uniqueness of invariant measures.

Ultimately, neither Haar nor von Neumann proved the existence of invariant measures on all locally compact groups. The first one to come up with a full proof was André Weil. This proof, however, was criticized for using the Axiom of Choice in the form of Tychonoff's Theorem. Later, Henri Cartan proved the existence of invariant measures on locally compact groups without the Axiom of Choice. Since then, several other people have also proved this theorem.

2 Definitions

Definition 1 A **topological group** G is a group as well as a topological space with the property that the mapping $(g_1, g_2) \mapsto g_1^{-1}g_2$ is continuous for all $g_1, g_2 \in G$. The multiplicative group of positive reals, for example, is a topological group since $(g_1, g_2) \mapsto \frac{g_2}{g_1}$ is continuous due to continuity of multiplication and nonzero division of real numbers.

Definition 2 A topological space X is said to be **locally compact** if for all $x \in X$, there is a compact set containing a neighborhood of x.

Definition 3 Let X be a topological space, and let $A \subset X$. Then A is **σ–bounded** if it is possible to find a sequence of compact sets $\{K_n\}_{n=1}^{\infty}$ with the property that $A \subset \bigcup_{n=1}^{\infty} K_n$.

Definition 4 A **left Haar measure** μ on a topological group G is a Radon measure which is invariant under left translation, i.e. $\mu(gB) = \mu(B)$ for all $g \in G$. A **right Haar measure** μ on a topological group G is a Radon measure which is invariant under right translation, i.e. $\mu(Bg) = \mu(B)$ for all $g \in G$.

Definition 5 A **content** λ is a set function that acts on the set of compact sets $\mathcal{C}$ that is finite, nonnegative, additive, subadditive, and monotone. A content induces an inner content and an outer measure. The inner content λ_* is defined by

$\lambda_*(A) = \sup\{\lambda(K) \mid K \in \mathcal{C}, K \subset A\}$. Let $\mathcal{O}$ denote the set of open sets. The outer measure μ_e is defined by $\mu_e(A) = \inf\{\lambda_*(O) \mid O \in \mathcal{O}, A \subset O\}$.

Definition 6 If μ_e is an outer measure, then a set A is said to be $\boldsymbol{\mu_e}$**–measurable** if for all sets B,

$$\mu_e(B) = \mu_e(A \cap B) + \mu_e(A^c \cap B).$$

3 Existence and Uniqueness

Theorem 1 On any locally compact group G, there exists a nonzero left Haar measure μ, and this Haar measure is unique up to a positive multiplicative constant of proportionality.

Proof The proof of this theorem relies on four lemmas.

Lemma 1 Let λ be a content, and let λ_* and μ_e be the inner content and outer measure, respectively, induced by λ. Then for all $O \in \mathcal{O}$ and for all $K \in \mathcal{C}$, $\lambda_*(O) = \mu_e(O)$ and $\mu_e(\mathrm{int}(K)) \leq \lambda(K) \leq \mu_e(K)$.

Proof For any $O \in \mathcal{O}$, it is clear that $\mu_e(O) \leq \lambda_*(O)$ since we can pick O as an open superset of O in the definition of μ_e. Now if $O' \in \mathcal{O}$ with $O \subset O'$, then $\lambda_*(O) \leq \lambda_*(O')$. Hence

$$\lambda_*(O) \leq \inf_{O'} \lambda_*(O') = \mu_e(O).$$

Therefore $\lambda_*(O) = \mu_e(O)$.

Now if $K \in \mathcal{C}$ and $O \in \mathcal{O}$ with $K \subset O$, $\lambda(K) \leq \lambda_*(O)$. Thus

$$\lambda(K) \leq \inf_{O} \lambda_*(O) = \mu_e(K).$$

If $K' \in \mathcal{C}$ with $K' \subset \mathrm{int}(K)$, then $\lambda(K') \leq \lambda(K)$, so

$$\mu_e(\mathrm{int}(K)) = \lambda_*(\mathrm{int}(K)) = \sup_{K'} \lambda(K') \leq \lambda(K). \qquad \blacksquare$$

Lemma 2 Let λ be a content, and let μ_e be the outer measure induced by λ. Then a σ–bounded set A is measurable with respect to μ_e if and only if for all $O \in \mathcal{O}$, $\mu_e(A \cap O) + \mu_e(A^c \cap O) \leq \mu_e(O)$.

Proof Let λ_* be the inner content induced by λ, let B be a σ–bounded set, and let $O \in \mathcal{O}$ satisfying $B \subset O$. Since

$$\lambda_*(O) = \mu_e(O) \geq \mu(A \cap O) + \mu_e(A^c \cap O) \geq \mu_e(A \cap B) + \mu_e(A^c \cap B),$$

$$\mu_e(B) = \inf_O \lambda_*(O) \geq \mu_e(A \cap B) + \mu_e(A^c \cap B).$$

The other direction and the converse follow from the definition of subadditivity and μ_e–measurability. ■

Lemma 3 Let μ_e be the outer measure induced by a content λ. Then the measure μ that satisfies $\mu(A) = \mu_e(A)$ for all Borel sets A is a regular Borel measure. μ is called the induced measure of λ.

Proof It suffices to show that each $K \in \mathcal{C}$ is μ_e–measurable. By Lemma 2, this would follow from showing that $\mu_e(O) \geq \mu_e(O \cap K) + \mu_e(O \cap K^c)$ for all $O \in \mathcal{O}$. Let $K' \in \mathcal{C}$ be a subset of $O \cap K^c$, and let $\widetilde{K} \in \mathcal{C}$ be a subset of $O \cap K'^c$. Clearly $O \cap K^c \in \mathcal{O}$ and $O \cap K'^c \in \mathcal{O}$. Because $K' \cap \widetilde{K} = \emptyset$ and $K' \cup \widetilde{K} \subset O$,

$$\mu_e(O) = \lambda_*(O) \geq \lambda(K' \cup \widetilde{K}) = \lambda(K') + \lambda(\widetilde{K}).$$

Thus

$$\mu_e(O) \geq \lambda(K') + \sup_{\widetilde{K}} \lambda(\widetilde{K}) = \lambda(K') + \lambda_*(O \cap K'^c)$$
$$= \lambda(K') + \mu_e(O \cap K'^c) \geq \lambda(K') + \mu_e(O \cap K).$$

Therefore,

$$\mu_e(O) \geq \mu_e(O \cap K) + \sup_{K'} \lambda(K') = \mu_e(O \cap K) + \lambda_*(O \cap K^c)$$
$$= \lambda(K') = \mu_e(O \cap K) + \mu_e(O \cap K^c).$$

Now it is necessary to show that $\mu(K)$ is finite. To do so, take $L \in \mathcal{C}$ with $K \subset \text{int}(L)$. Then

$$\mu(K) = \mu_e(K) \leq \mu_e(\text{int}(L)) \leq \lambda(L) < \infty.$$

Finally, regularity follows from

$$\mu(K) = \mu_e(K) = \inf_O\{\lambda_*(O) \mid K \subset O, O \in \mathcal{O}\} = \inf_O\{\mu_e(O) \mid K \subset O, O \in \mathcal{O}\}$$
$$= \inf_O\{\mu(O) \mid K \subset O, O \in \mathcal{O}\}. \qquad \blacksquare$$

Lemma 4 Let Ω be a measurable space and let $h : \Omega \to \Omega$ be a homeomorphism. Let λ and κ be contents on Ω such that for all $K \in \mathcal{C}$, $\lambda(h(K)) = \kappa(K)$. Suppose that μ and ν are the induced measures of λ and κ, respectively. Then $\mu(h(A)) = \nu(A)$ for any Borel measurable set $A \in \Omega$.

Proof Let λ_* and κ_* be the inner contents induced by λ and κ, respectively, and let μ_e and ν_e be their respective outer measures. If $O \in \mathcal{O}$, then

$$\{\kappa(K) \mid K \subset O, K \in \mathcal{C}\} = \{\lambda(h(K)) \mid K \subset O, K \in \mathcal{C}\}$$
$$= \{\lambda(A) \mid A = h(K), K \subset O, K \in \mathcal{C}\}$$
$$= \{\lambda(A) \mid h^{-1}(A) \subset O, h^{-1}(A) \in \mathcal{C}\}$$
$$= \{\lambda(A) \mid A \subset h(O), A \in \mathcal{C}\}.$$

Thus $\kappa_*(O) = \lambda_*(h(O))$. Let B a σ–bounded set. Then

$$\{\kappa_*(O) \mid B \subset O, O \in \mathcal{O}\} = \{\lambda_*(h(O)) \mid B \subset O, O \in \mathcal{O}\}$$
$$= \{\lambda_*(C) \mid C = h(O), B \subset O, O \in \mathcal{O}\}$$
$$= \{\lambda_*(C) \mid h^{-1}(C) \mid h^{-1}(C) \subset B, h^{-1}(C) \in \mathcal{O}\}$$
$$= \{\lambda_*(C) \mid C \subset h(B), C \in \mathcal{O}\}.$$

Thus $\nu_e(B) = \mu_e(h(B))$. By the result of Lemma 3, if A is any Borel set, then $\mu(h(A)) = \nu(A)$. $\blacksquare$

Because of Lemma 4, one must simply find a content λ on G which is invariant under left translation to demonstrate existence. By Lemma 1, the induced measure of λ will be nonzero.

Let $A \subset G$ be a bounded set, and let $B \subset G$ be a set with nonempty interior. Then let $A : B$ denote the lowest positive integer n such that there exists a set $\{g_j\}_{j=1}^n \subset G$ with the property that $A \subset \bigcup_{j=1}^n g_j B$. Now let $A \in \mathcal{C}$ be a set with nonempty interior. Let $\mathcal{N}$ denote the set of all neighborhoods of the identity element of G. Fix $O \in \mathcal{N}$. Now define

$$\lambda_O(K) = \frac{K : O}{A : O}$$

for $K \in \mathcal{C}$. Clearly $\lambda_O(K)$ satisfies $0 \le \lambda_O(K) \le K : A$. $\lambda_O(K)$ clearly satisfies all the properties of a content other than additivity.

For each $K \in \mathcal{C}$, consider the interval $I_K = [0, K : A]$, and let $\Xi = \prod I_K$. By Tychonoff's Theorem, Ξ is compact. Ξ consists of points that are the direct products of functions ϕ acting on $\mathcal{C}$ with the property that $0 \le \phi(K) \le K : A$. $\lambda_O \in \Xi$ for all $O \in \mathcal{N}$.

Now define

$$\Lambda(O) = \{\lambda_{O'} \mid O' \subset O, O' \in \mathcal{N}\}$$

given $O \in \mathcal{N}$. If $\{O_j\}_{j=1}^n \subset \mathcal{N}$, then

$$\Lambda\left(\bigcap_{j=1}^n O_j\right) \subset \bigcap_{j=1}^n \Lambda(O_j).$$

Clearly $\Lambda\big(\bigcap_{j=1}^n O_j\big)$ is nonempty. Since Ξ is compact, there is some point in the intersection of the closures of all the Λs

$$\lambda \in \bigcap_O \{\overline{\Lambda(O)} \mid O \in \mathcal{N}\}.$$

It is now necessary to prove that λ is in fact a content. For any $K \in \mathcal{C}$, $\lambda(K)$ is finite and nonnegative since $0 \le \lambda(K) \le K : A < \infty$. To prove monotonicity and subadditivity, let $\xi_K(\phi) = \phi(K)$. Then ξ_K is a continuous function. Thus if K_1 and K_2 are compact sets, then

$$\Theta = \{\phi \mid \phi(K_1) \le \phi(K_2)\} \subset \Xi$$

is closed. Then let $K_1 \subset K_2$ and $O \in \mathcal{N}$. Then $\lambda_O \in \Theta$, and hence $\Lambda(O) \subset \Theta$. Since Θ is closed, $\lambda \in \overline{\Lambda(O)} \subset \Theta$, which implies that λ is monotone and subadditive.

To prove additivity, first note the restricted additivity of λ_O. Let gO be a left translation of O, and fix $K_1, K_2 \in \mathcal{C}$ so that $K_1O^{-1} \cap K_2O^{-1} = 0$. If $K_1 \cap gO \neq 0$, then $g \in K_1O^{-1}$, and if $K_2 \cap gO \neq \emptyset$, then $g \in K_2O^{-1}$. Thus there are no left translations of O that do not intersect either K_1 or K_2, and so λ_O has additivity given that $K_1O^{-1} \cap K_2O^{-1} = \emptyset$. Let $K_1, K_2 \in \mathcal{C}$ with $K_1 \cap K_2 = \emptyset$. Then there is some $O \in \mathcal{N}$ satisfying $K_1O^{-1} \cap K_2O^{-1} = \emptyset$. If $O' \in \mathcal{N}$ and $O' \subset O$, then $K_1O'^{-1} \cap K_2O'^{-1} = \emptyset$ as well. Thus $\lambda_{O'}(K_1 \cup K_2) = \lambda_{O'}(K_1) + \lambda_{O'}(K_2)$. Then if $O' \subset O$,

$$\lambda_{O'} \in \Theta' = \{\phi \mid \phi(K_1 \cup K_2) = \phi(K_1) + \phi(K_2)\}.$$

Thus λ is additive. This establishes the existence of a Haar measure on any locally compact group.

To establish uniqueness, let μ be a left Haar measures, and consider a nonnegative continuous function f on a locally compact group G that is not identically zero. Since $\int_G f\ d\mu > 0$, we may assume that $\int_G f\ d\mu = 1$. Let us write

$$\Psi(g) = \int_G f(xg^{-1})\ d\mu(x),$$

where $g \in G$. Then $\Psi : G \to \mathbb{R}^+$ is a continuous function and also a homomorphism. Now select a continuous function h on G and consider the convolution

$$(f * h)(g) = \int_G f(x)h(x^{-1}g)\ d\mu(x) = \int_G f(gx)h(x^{-1})\ d\mu(x).$$

By the definition of Ψ and $\int_G f\ d\mu = 1$,

$$\int_G h(x)\ d\mu(x) = \int_G h(x^{-1})\Psi(x^{-1})\ d\mu(x).$$

A right translation of h gives

$$\begin{aligned}\int_G h(xg^{-1})\ d\mu(x) &= \int_G h(x^{-1}g^{-1})\Psi(x^{-1})\ d\mu(x) \\ &= \Psi(g)\int_G h((gx)^{-1})\Psi((gx)^{-1})\ d\mu(x) \\ &= \Psi(g)\int_G h(x^{-1})\Psi(x^{-1})\ d\mu(x).\end{aligned}$$

Thus

$$\Psi(g) = \frac{\int_G h(xg^{-1})\, d\mu(x)}{\int_G h(x)\, d\mu(x)}.$$

Now let υ and ϕ be two continuous functions on G, and let Ψ be defined as above. Also, let ν be another left Haar measure. Then

$$\begin{aligned}
\int_G \upsilon(x)\, d\mu(x) \int_G \phi(y)\, d\nu(y) &= \int_G \int_G \upsilon(x)\, d\mu(x)\phi(y)\, d\nu(y) \\
&= \int_G \int_G \upsilon(xy)\, d\mu(x)\Psi(y)\phi(y)\, d\nu(y) \\
&= \int_G \int_G \upsilon(xy)\phi(y)\Psi(y)\, d\nu(y)\, d\mu(x) \\
&= \int_G \int_G \upsilon(y)\phi(x^{-1}y)\Psi(x^{-1}y)\, d\nu(y)\, d\mu(x) \\
&= \int_G \int_G \phi((y^{-1}x)^{-1})\Psi((y^{-1}x)^{-1})\, d\mu(x)\upsilon(y)\, d\nu(y) \\
&= \int_G \int_G \phi(x^{-1})\Psi(x^{-1})\, d\mu(x)\upsilon(y)\, d\nu(y) \\
&= \int_G \phi(x)\, d\mu(x) \int_G \upsilon(y)\, d\nu(y).
\end{aligned}$$

Thus $\int_G \upsilon\ d\mu \int_G \phi\ d\nu = \int_G \phi\ d\mu \int_G \upsilon\ d\nu$. Now letting υ be a positive continuous function and setting

$$c = \frac{\int_G \upsilon\, d\nu}{\int_G \upsilon\, d\mu}$$

gives $\int_G \phi\, d\nu = c \int_G \phi\, d\mu$. ■

4 Bibliography

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