Problem: Find all integer solutions to the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}.$$

Solution: We will show that all solutions are of the form:

$$\begin{aligned} x &= k \cdot a \cdot (a+b) \\ y &= k \cdot b \cdot (a+b) \\ z &= k \cdot a \cdot b, \end{aligned}$$

where k, a, and b are arbitrary non-zero integers, $a + b \neq 0$.

We can rewrite the given equation as

$$z = \frac{xy}{x+y},$$

so it suffices to find all pairs of integers x, y such that $x + y \mid xy$.

Lemma: Let r and s be relatively prime positive integers. Then $r \pm s$ and rs are relatively prime.

Proof: Suppose not; then there exists an integer $k > 1$ such that $k \mid r \pm s$ and $k \mid rs$. We have

$$k \mid r \pm s \Rightarrow k \mid r^2 \pm rs \Rightarrow k \mid r^2.$$

Similarly, $k \mid s^2$.

Let $p > 1$ be any prime factor of k, so that $p \mid r^2$ and $p \mid s^2$. Now simply note that for any positive integer m, $p \nmid m \Rightarrow p \nmid m^2$. Hence, we must have $p \mid r$ and $p \mid s$, which contradicts our assumption that r and s are relatively prime. So $r \pm s$ and rs must be relatively prime. //

Let x and y be integers such that $x + y \mid xy$. We clearly cannot have $x + y = 0$. If $|x + y| = 1$, then we have the solution sets

$$\begin{aligned} x &= c \\ y &= -(c+1) \\ z &= c \cdot (c+1) \end{aligned}$$

and

$$\begin{aligned} x &= c+1 \\ y &= -c \\ z &= -c \cdot (c+1), \end{aligned}$$

where c is an integer, $c \neq 0, 1$.

We now assume $|x + y| > 1$. Let $n = \gcd(|x|, |y|)$. Suppose that $|x|$ and $|y|$ are relatively prime. Then by the lemma, $|x + y|$ and $|xy|$ are relatively prime, a contradiction, since $x + y \mid xy$ and $|x + y| > 1$.

Hence, $n > 1$. Choose integers a, b such that $x = na$ and $y = nb$. This gives:

$$\begin{aligned} & (na + nb) \mid (na) \cdot (nb) \\ \Leftrightarrow\ & n \cdot (a + b) \mid n^2 \cdot a \cdot b \\ \Leftrightarrow\ & a + b \mid n \cdot a \cdot b. \end{aligned}$$

Since $n = \gcd(|x|, |y|)$, $|a|$ and $|b|$ must be relatively prime. So by the lemma, the above holds if and only if $a + b \mid n$. Put $n = k \cdot (a + b)$. This gives the solution set:

$$\begin{aligned} x &= k \cdot a \cdot (a + b) \\ y &= k \cdot b \cdot (a + b) \\ z &= k \cdot a \cdot b. \end{aligned}$$

Finally, note that the previous two solution sets are contained within this one (for the first set, take $k = -1,\ a = c,\ b = -(c + 1)$; for the second set, take $k = 1,\ a = c + 1,\ b = -c$). Hence, this is the entire family of solutions, as desired. ■

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