Proof of the Area of a Circle Formula $A = \pi r^2$

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Theorem 1. *The area of a circle with radius r is πr^2.*

Proof: The equation of a circle centered at the origin is

$$x^2 + y^2 = r^2,$$

where r is the radius. We write y in terms of the variable x and the constant r:

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\frac{y}{r} = \sqrt{1 - \frac{x^2}{r^2}}$$

$$y = r\sqrt{1 - \frac{x^2}{r^2}}$$

By symmetry, the area of a circle centered at the origin is four times the area of the circle between $(0, 0)$ and $(r, 0)$ above the x-axis. We can integrate to find the area (A):

$$A = 4r \int_0^r \sqrt{1 - \frac{x^2}{r^2}}\, dx$$

To evaluate the antiderivative of $\sqrt{1 - \frac{x^2}{r^2}}$, we make the substitutions:

$$x = r \sin\theta$$

$$\theta = \arcsin\frac{x}{r}$$

$$dx = r\cos\theta\, d\theta$$

Thus, our integral becomes:

$$A = 4r \int_0^r \sqrt{1 - \frac{x^2}{r^2}}\, dx = 4r \int_0^{\pi/2} r\sqrt{1 - \sin^2\theta} \cos\theta \, d\theta$$

We can use the trigonometric identity $1 - \sin^2\theta = \cos^2\theta$:

$$A = 4r \int_0^{\pi/2} r\sqrt{1 - \sin^2\theta} \cos\theta \, d\theta = 4r^2 \int_0^{\pi/2} \cos^2\theta \, d\theta$$

We then apply $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$:

$$\begin{aligned} 4r^2 \int_0^{\pi/2} \cos^2\theta \, d\theta &= 4r^2 \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta)\, d\theta \\ &= 2r^2\theta\Big|_0^{\pi/2} + 2r^2 \int_0^{\pi/2} \cos 2\theta \, d\theta \\ &= \pi r^2 + 2r^2(\sin 2\theta)\Big|_0^{\pi/2} \\ &= \pi r^2 \end{aligned}$$

Thus, the area of a circle with radius r is πr^2. ■